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Stochastic modeling of stress fields in geotechnical problems with discrete media

Modélisation stochastique des contraintes pour les problèmes géotechniques en milieux discrets

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ABSTRACT: An important class of soil mechanics problems is the determination of the stresses caused in a soil body by the application of a certain load. The theory of elasticity provides solutions when the soil is supposed to be continuous, isotropic and linearly elastic: the stress field is obtained from the solution of a boundary value problem. In contrast, when the medium is not a continuous body but a dense packing of discrete elastic particles, interparticle forces are the responsible for supporting the loads. The equivalent stress of any particle in the packing can be estimated from the forces keeping it in static equilibrium, but the obtained value depends on the specific realization of the packing and is most often different to that predicted by the solution to the corresponding boundary value problem. Statistical mechanics can be used to anticipate the statistical distributions of stress values and to relate the associated parameters to continuum mechanics solutions. The theory has been validated through massive numerical simulation with the discrete element method. Accordingly, in the case of discrete media, the deterministic model for the stress field in the soil can be replaced by a stochastic one, which gives the probability of finding a certain level of stress at any position. This model is useful when the spatial scale of interest is comparable to the size of the particles (e.g. grain-level analysis, altered zones,). When the number of intervening particles is large, the average stress matches the value predicted by continuum mechanics approaches.

RÉSUMÉ: Une classe importante de problèmes en mécanique des sols vers sur la détermination des contraintes provoquées dans un corps de sol par l'application d'une certaine charge. La théorie de l'élasticité fournit des solutions lorsque le sol est censé être continu, isotrope et linéairement élastique: les contraintes sont obtenues de la solution d'un problème de valeur limite. En revanche, lorsque le milieu n'est pas un corps continu, mais un empliment compact de particules élastiques discrètes, les forces interparticulaires sont les responsables du support des charges. La contrainte équivalente de n'importe quelle particule dans l'empliment peut être obtenue à partir des forces qui la maintenant en équilibre statique, mais valeur obtenue dépend de la réalisation spécifique de l'empilment et elle est normalement différente de celle-là prédite par la solution du problème de valeur limite correspondant . Nous avons predites les distributions statistiques de ces valeurs, avons lié les paramètres associés aux solutions classiques et avons validé la théorie au travers d'une simulation numérique massive avec la méthode des éléments discrets. Ce faisant, pour les empilements compacts, le modèle déterministe des contraintes dans le sol doit être remplacé par un modèle stochastique qui donne la probabilité de trouver un certain contrainte en fonction de la position. Ce modèle est utile lorsque l'echelle spatial d'intérêt est comparable à la taille des particules (par exemple, une analyse au niveau du grain, des zones modifiées, etc.). Lorsque le nombre de particules intervenant est important, la contrainte moyenne sera cella prédite par la théorie de l'elasticité.

Keywords: Theoretical analysis; Statistical Analysis; Discrete-element modelling; Elasticity

1 INTRODUCTION

Some seminal problems in geotechnics were solved under the assumptions that the soil is a continuous and elastic body. The solution of a boundary value problem gives the stress and strain fields, which are valid as long as the behavior of the soil remains elastic, deformations are small and the medium can be treated as a continuous body. A great part of advanced modeling in geotechnics concerns with the non-elastic behavior of the soil, but the discrete nature of some soils cannot always be ignored. This could be the case of grain scale analyses, rockfills, altered rock masses, heterogeneous terrains, etc. In such circumstances continuum based approaches may fail and the discrete nature of the medium must be included in the model.

In this work a stochastic approach is presented. It applies when the size of particles or heterogeneities is comparable to the spatial scale of interest. On that scale, there is no smooth stress field, but a set of interparticle forces keeping the system in equilibrium. These forces can be transformed into equivalent stresses, but the resulting field is heterogeneous and packing-dependent. We study this stress field with a stochastic model that gives the probability of finding a value of stress at a given position.

The methodology is illustrated for the next Boussinesq-like problem: a vertical and finite surface load acts on the surface of a 2D dense packing of almost equally-sized elastic particles. The statistical distribution of vertical stress is anticipated by following statistical mechanics approaches. The model has been then validated through massive numerical simulation with the discrete element method.

The work is presented as follows: motivation (including a thought experiment that introduces the theoretical bases of the model), methodology (continuum and discrete approaches, homogenization techniques and statistical mechanics), results, discussion and conclusion.

2 MOTIVATION

2.1 The problem

So far, the most common way of determining the stresses caused in a discrete body by the application of a certain load has been:

1 - Assume that the soil is a continuous body and determine equivalent elastic moduli and unitary weight.

2 - Solve the boundary value problem and get the value of the stress at the control point.

In a continuous an elastic half-space the load is supported by a stress field that is illustratively represented in Figure 1.

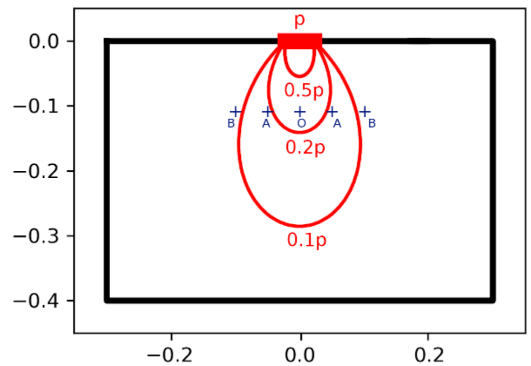

Figure 1: Vertical stress field caused by a surface load p acting on a continuous and elastic half-space. The stress bulbs connect all the points having the same value of the vertical stress. O, A and B are the control points used in section 3.

An alternative to continuum based approaches is given by discrete models. Numerical and laboratory experiments show how the load is supported by a system or interparticle forces (Radjai et al., 1996; Majmudar and Behringer, 2005). A procedure based on discrete approaches could be:

1- Randomly generate a packing of the system of discrete particles in equilibrium with the boundary conditions and body forces.

2- Find the particle that is closest to the control point.

3- Consider the interparticle forces that maintain that particle in static equilibrium.

4- Get the stress at the control point. This can be done either by solving a particle-level boundary value problem and getting the exact value at the control point or by applying homogenization techniques and assigning the average value of the stress to the control point. The latter approach is much simpler and it will be followed herein.

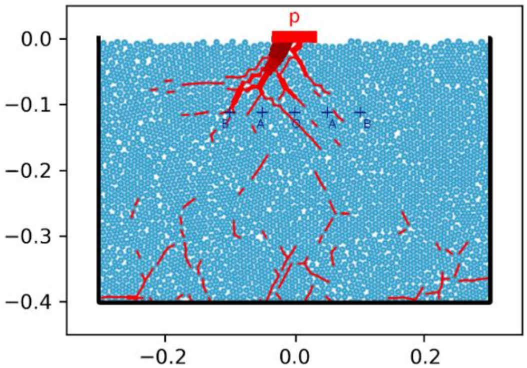

Figure 2: A system of force-chains supports the surface load p that acts on the surface of a dense packing of discrete particles. O, A and B are the control points used in section 3.

Continuum and discrete approaches provide values of the stress at the control point that are most of the time different from each other. The former is deterministic and always gives the same value, and the latter gives a value that is packing-dependent. The second approach discloses the stochastic nature of the force-chains map. However, the packing realization can be massively repeated in order to generate a statistical sample from which anticipate the distributions. Then two questions would arise:

1-Should the mean value of the stress measured in many different realizations of the experiment correspond to the value obtained from the continuum based approach?

2 - If yes, what are the plausible values of the stress that can be found at the control point?

In this work, questions 1 and 2 are solved for a specific seminal problem in geotechnics. Additionally, this research helps to establish the spatial scale on which the continuum approach fails. But before answering this questions, a thought experiment illustrate how these questions can be addressed.

2.2 Thought experiment: A wind turbine founded on 5 piles.

Let be a wind turbine founded on 5 steel pipe piles in a 2D world. The maximum lateral design force is 5 force units (FU). All the piles are equal and should equally work, but the heterogeneities in the soil and some variations of the driving process have led to a different transfer of load to each of the piles. For the lack of simplicity, the load that each pile is actually resisting is assumed to be a discrete random variable that is uniformly distributed in the interval 0 to 5 FU. Each pile could work independently of what the others do, provided that the total lateral force is resisted by the group. This is the only constraint of the problem, since neither the moment balance nor vertical loads are considered. What should be the maximum lateral load capacity of the piles? What is the risk of pile failure it was below 5 FU?

In order to answer these questions, we must analyze the sample space of this thought experiment, which is made of 126 different cases (out of the 7726 possible 5-element variations of 6-load levels with repetition allowed). Among the 126 cases forming the class in which the total lateral force is 5 FU, there is one (and only one) case in which all the piles equally share the load. This load sharing is denoted as (1 1 1 1 1) and the subclass is referred to as the optimistic because there is no pile resisting more than 1 FU. In contrast there are 5 possible cases in which a pile resists alone the whole lateral load: (5 0 0 0 0), (0 5 0 0 0), (0 0 5 0 0), (0 0 0 5 0) and (0 0 0 0 5). These cases form the pessimistic subclass, which, in the absence of any further information, would be expected to be 5 times more likely than the optimistic. Following this reasoning, the most likely subclasses are those in which there are 2 piles unloaded and 2 of the remaining 3 are equally loaded (e.g.: (1 3 1 0 0) or (2 0 2 0 1)). These most likely subclasses include 30 cases each, and both are thus 30 times more likely than that the optimistic one and 6 times more likely than the pessimistic one.

(5 0 0 0 0)	(0 1 4 0 0)
(0 5 0 0 0)	(0 0 4 1 0)
(0 0 5 0 0)	(0 0 4 0 1)
(0 0 0 5 0)	(1 0 0 4 0)
(0 0 0 0 5)	(0 1 0 4 0)
(4 1 0 0 0)	(0 0 1 4 0)
(4 0 1 0 0)	(0 0 0 4 1)
(4 0 0 1 0)	(1 0 0 0 4)
(4 0 0 0 1)	(0 1 0 0 4)
(1 4 0 0 0)	(0 0 1 0 4)
(0 4 1 0 0)	(0 0 0 1 4)
(0 4 0 1 0)	(3 2 0 0 0)
(0 4 0 0 1)	(3 0 2 0 0)
(1 0 4 0 0)	...

Figure 3: Thought experiment: The wind causes on the wind turbine a load of 5 FU in the direction perpendicular to the drawing. The list of possible load sharings among the piles (i.e. those compatible to the problem constraints) is partially shown on the right.

Table 1: Subclasses definition according to the number of piles loaded at each force level.

Subclass \ F(FU)	**0**	**1**	**2**	**3**	**4**	**5**	**Prob.**
Pessimistic	4	0	0	0	0	1	5/126
	3	1	0	0	1	0	20/126
	3	0	1	1	0	0	20/126
Most likely 1	2	2	0	1	0	0	30/126
Most likely 2	2	1	2	0	0	0	30/126
	1	3	1	0	0	0	20/126
Optimistic	0	5	0	0	0	0	1/126

Some practical conclusions:

- The number of cases satisfying the only constraint of the problem (i.e. total lateral load equal to 5 FU) is 126. If a moment balance was imposed, then the number of cases would be drastically reduced to 6 (only symmetric distributions): (0 0 5 0 0), (0 1 3 1 0), (1 0 3 0 1), (0 2 1 2 0), (2 0 1 0 2) and (1 1 1 1 1).
- The case (1 1 1 1 1), belonging to the optimistic subclass, is as likely as the (1 3 1 0 0), belonging to one of the most likely subclasses. However, the optimistic subclass is much less likely (30 times) than this most likely subclass.
- Probably the case (1 1 1 1 1) would attract our attention much more than, say, the (1 3 1 0 0), unless it matched a number that we recognize (car's plate, phone number…). This is because of its symmetries and equalities. In the absence of these features (which actually identify with additional constraints), we would not appreciate to much difference between cases belonging to the same subclass.
-In this thought experiment, thinking that the piles are resisting 1 FU each is too optimistic (1% likely); thinking that one of piles is withstanding 5 FU could be too conservative (it only happens in 4% of cases). For designing purposes, the important thing is the subclass, not the case.

In this example there are only 5 piles. If the number was much larger, we would conclude that in the most probable class, the distribution of load among the piles would follow an exponential distribution. Then the probability that a pile supports, say, twice the mean force, would be 13.5%.

3 METHODS

A stochastic model has been proposed for the next classical problem: a 2D half-space that is a dense packing of discrete elastic particles is objected to the action of the gravity (g = 9.81 m/s^2) and of a surface load p = 44.4 kPa/m that acts over a finite area of width 2a = 0.045 m. The objective is to know which is the vertical stress at three given points, O, A and B, which are located 0.11 m below the surface and which horizontally separate from the center of the load a distance of 0.00, 0.08 and 0.15 m, respectively (see Figure 1 and Figure 2).

3.1 The continuum based approach

If the medium is treated as a continuous body, then the solution of the corresponding boundary value problem gives a vertical stress σ_{zz} (kPa) equal to (Verruijt, 2006):

$$\sigma_{zz} = \sigma_{\mathrm{g},zz} + \sigma_{\mathrm{p},zz} \quad (1)$$

$$\sigma_{\mathrm{g},zz} = \rho_s \mathrm{g}\,(1-n)z \quad (2)$$

$$\sigma_{\mathrm{p},zz} = \frac{p}{\pi}[(\theta_1-\theta_2)+\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1] \quad (3)$$

Where ρ_s (kg/m^3) is the density of the solid particles, g (m/s^2) is the gravitational acceleration, n (-) is the porosity, z (m) is the depth, p (kPa) is the surface load and $\theta_1 = \arctan(X_1 - x)/z$ and $\theta_2 = \arctan(X_2 - x)/z$, being X_1 (m) and X_2 (m) the left and right limits of the surface load.

3.2 A discrete approach

3.2.1 The discrete element method

The discrete element method DEM (Cundall and Strack, 1979), can be used to generate packings of discrete media in static equilibrium compatible to the load and boundary conditions. The DEM computes the motion of a set of virtual solid particles after considering mutual interactions. We followed a simple formulation in which particles are spherical and the mechanical contact is represented by means of spring-dashpot interactions. Then the normal interaction force is given by:

$$\boldsymbol{F}_{\mathrm{n},ij} = -k_{\mathrm{n},ij}\boldsymbol{\delta}_{\mathrm{n},ij} \quad (4)$$

Where $k_{\mathrm{n},ij}$ (kN/m) is the normal contact stiffness, $k_{\mathrm{n},ij} = 2ER_iR_j/(R_i+R_j)$, $\delta_{ij} = (R_i+R_j) - r_{ij}$ (m) is the overlap, R_i and R_j (m) are the radii of the particles and r_{ij} (m) is the distance between them.

Tangential forces are produced in opposition to incremental lateral displacements. These forces are limited by the value of normal forces and friction coefficients:

$$\boldsymbol{F}_{\mathrm{s},ij} = -\min(-k_{\mathrm{s},ij}u_{\mathrm{n},ij}, \tan\emptyset\, F_{\mathrm{n},ij})\,\boldsymbol{u}_{ij}/u_{ij} \quad (5)$$

Where $k_{\mathrm{s},ij}$ (kN/m) is the tangential contact stiffness, u_{ij} (m) is the lateral displacement between the two particles previously in contact and $\emptyset$ is the interparticle friction angle.

3.2.2 Numerical experiments

Around one thousand numerical experiments were performed with YADE-DEM (Smilauer et al., 2015). 5000 particles were randomly poured within a 1.0 m wide domain and the system was let to almost completely dissipate its kinetic energy. The particles, whose properties are shown in Table 2, followed a quasi-uniform particle size distribution in the interval $[D-\Delta D, D+\Delta D]$. The surface load was applied by gently and vertically (downwards) moving a rigid body until the total vertical force on this rigid element was equal to $2ap$.

Table 2. Properties of particles used in numerical simulations

Property		Value	Unit
Number of particles	N	5,000	-
Simulation width	L	1.0	m
Diameter dispersion	$\Delta D/D$	0.05	-
Young's modulus	E	$1.0\ 10^{10}$	Pa
Material density	ρ_s	$2.6\ 10^3$	kg/m^3
Interparticle friction	φ	0	rad

3.2.3 Stress homogenization

The volumetric average of the stress field within a domain extracted from a continuous body that is in static equilibrium can obtained from the tensor product of the forces on the external boundaries by their application positions (Bagi, 1996):

$$\langle\sigma_{ij}\rangle = \frac{1}{V}\int_V \sigma_{ij}dV = \frac{1}{V}\sum_l x_i F_j^l \quad (6)$$

Where V (m^3) is the volume of the domain.

As a derivation of formula (6), the extensive stress Σ_{ij} (Jul) is defined as:

$$\Sigma_{ij} = \langle\sigma_{ij}\rangle \mathrm{V} = \sum_l x_i F_j^l \quad (7)$$

An important property of the extensive stress is that it is additive: the total extensive stress of a composite body can be obtained from the sum of the extensive stress of its constituents:

$$\Sigma_{ij}^{A+B} = \Sigma_{ij}^{A} + \Sigma_{ij}^{B} \tag{8}$$

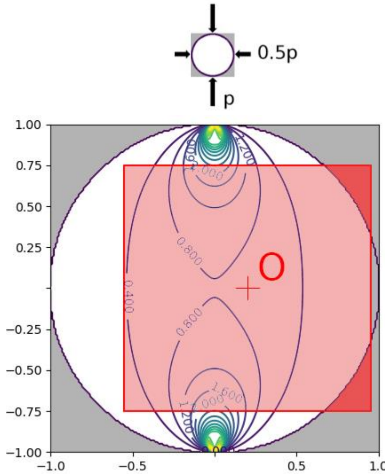

Figure 4: Illustrative example of inner (vertical) stress field caused by two sets of diametrically opposed line forces in a cylindrical particle in static equilibrium The reddish region represents the control volume around the control point O.

This homogenization technique can be illustratively applied to the particle shown in Figure 4. This cylindrical particle of radius R = 1.0 m is in static equilibrium under the action of two sets of diametrically opposed line forces: vertical forces of value p = 1.0 N/m and horizontal forces of value 0.5p. The Voronoi cell in this case is a square of side 2R. The results of the homogenization are summarized in Table 3.

This technique was also used to estimate the extensive stress of the closest particle to the control point P in the packings generated with DEM. The control volume V_c in all the packings was a square region of side R = 0.005 m centered at the control positions. The extensive stress of the control volume was supposed to be $\Sigma_{ij}^{c} = \Sigma_{ij}^{P}\, V_c/V_P$, being V_P the volume of the particle and the associated void space and $\Sigma_{ij}^{P} = \sum_l x_i F_j^l$ the extensive stress of the particle obtained from external forces.

Table 3. Stress homogenization of the example in Figure 4)

Property		**Value**	**Unit**
Total volume	V	4	m^3
Particle volume	V	π	m^3
Porosity	n	0.21	-
Vertical extensive stress	Σ_{zz}	2.0	Jul
Average vertical stress	σ_{zz}	0.5	Pa
Horizontal extensive stress	Σ_{xx}	1.0	Jul
Average horizontal stress	σ_{xx}	0.25	Pa
Shear extensive stress	Σ_{xy}	0.0	Jul
Average shear stress	σ_{xy}	0.0	Pa

3.3 Statistical mechanics

Statistical mechanics (Balescu, 1975) is the branch of physics that deals with systems made of a large number of constituents. Since first model (Edwards & Oakeshott, 1989), several statistical mechanics approaches to granular media have been proposed. We have followed that based on the extensive stress (Edwards, 2005; Henkes et al., 2007; Tejada, 2014). The basic idea is that among the ways of packing a granular system, there is a class that is compatible to the constrains imposed by the associated elastic boundary problem (boundary conditions and external loads). This class still contains an enormous number of solutions and, in the absence of further information, there is no a priori reason for favoring one of these more than any other (principle of equal a priori probabilities). If a subdomain of the packing (made of a solid particle plus the associated void space) is extracted, it may take different values of the extensive stress. If N realizations of the packing were added up, the total extensive stress of the sum would be equal to the sum of the extensive stress got in each realization. Then next hypotheses are considered:

- Particles may take any positive value of the extensive stress with the same probability. Negative values are not allowed because interactions are not cohesive.

- Any packing can be tessellated with a Voronoi diagram so that cells are made of a particle plus an associated volume space (Figure 4). The volumetric average of the stress field within a control volume within a Voronoi cell (region shaded in red in Figure 4) is equal to the volumetric average of the stress within the whole cell..

- The mean value of the extensive stress obtained by generating many packings of the system satisfying the constraints of the problem must be equal to the value obtained by multiplying the volume by the value of stress given by the solution of the corresponding boundary problem.

Under these hypotheses, it can be proved (Tejada, 2018) that the distribution of vertical extensive stresses that maximizes the entropy (i.e. the subclass of solutions that can be found in a larger number of cases) is an exponential distribution of pdf:

$$f_{(\Sigma_{zz})} = \frac{1}{\mu_{zz}} e^{-\frac{\Sigma_{zz}}{\mu_{zz}}} \tag{7}$$

Where $\mu_{zz} = \sigma_{zz} V_c$, V_c (m^3) is the control volume and $\sigma_{zz} = \sigma_{g,zz} + \sigma_{p,zz}$ (Equations 1-3). This expression also applies for the horizontal stress (with a different mean) but not for shear stresses. This is because the extensive shear stress can be positive or negative.

4 RESULTS

4.1 *The mean value (continuum based approach)*

Table 4: Expected stress and extensive stress at the control points (V_c = 0.005x0.005x1.0 m^3).

Control point	$\sigma_{g,zz}$ **(kPa)**	$\sigma_{p,zz}$ **(kPa)**	σ_{zz} **(kPa)**	Σ_{zz} **(Jul)**
O	*2.3*	11.3	13.6	0.34
A	*2.3*	5.0	7.4	0.18
B	*2.3*	1.5	3.8	0.09

Assuming an average porosity of n = 0.17 (the value obtained in DEM simulations) and the material density included in Table 2, the values of the vertical stresses expected at control points O, A and B are shown in Table 4.

4.2 *The statistical distribution (discrete based approach)*

The obtained height of the packing after pouring the particles within the domain was H = 0.46 ± 0.01 m, which corresponds to an average porosity of n = 0.17 ± 0.01.

A comparison between the expected (exponential) and the obtained distribution at the 3 control points is shown in Figure 5

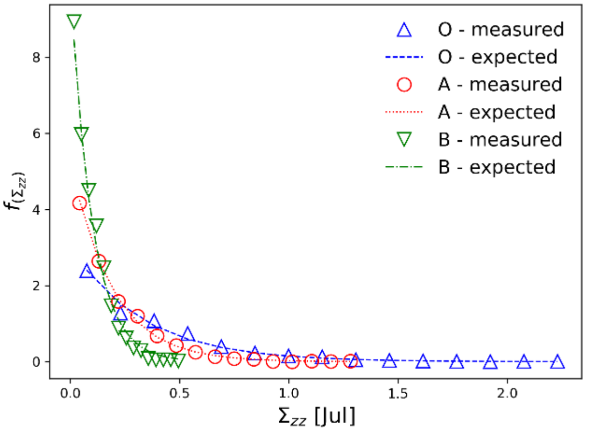

Figure 5: Expected (exponential) and measured statistical distribution of the vertical extensive stress at three different points in the sample.

5 DISCUSSION

Massive simulation with the DEM has proven that the measured statistical distribution of extensive stress matches that predicted in by statistical mechanics. Therefore, the problem of determining the stress field at a given position in discrete media can be solved with a stochastic model described by an exponential distribution. The procedure would therefore be:

1 – Solve the associated boundary value problem and get the value of the vertical stress at the point of interest σ_{zz}.

2 - Multiply that field by the volume of the particle plus its corresponding void space V_c to get a point estimation of the vertical extensive stress $E[\Sigma_{zz}] = V_c \sigma_{zz}$.

3 – Use the exponential distribution to get interval estimation of the vertical extensive stress: $\Sigma_{zz} \sim Exp(\lambda)$ with $\lambda = {}^{1}/_{\mathrm{V}_c \sigma_{zz}}$.

Therefore, the two questions posed in Section 2 have been answered: 1) the mean value of the stress over the sample of realizations of the experiment corresponds to the value obtained under the assumption of continuity and 2) the statistical distribution follows an exponential function. But, what is the spatial scale on which the stochastic model is unneeded? On a larger spatial scale, the number of particles intervening may be so large that, because of the law of large number, the average value of the stress would precisely be the value predicted by continuum approaches. Let's think again on the continuum based solution to the problem studied here. Let's imagine that there are N particles within a small volume V whose extensive vertical stress follows and exponential distribution of parameter $\lambda = 1/(V_c \sigma_{zz})$, being σ_{zz} the stress within that volume that is predicted by continuum mechanics approaches and $V_c = V/N$. If $= \Sigma_{ij}^{m} \sim \mathrm{Exp}\big(N/(V\sigma_{zz})\big)$, then $\Sigma_{zz} = \sum_{m=1}^{N} \Sigma_{ij}^{m} \sim \mathrm{Erlang}\big(N, N/(V\sigma_{zz})\big)$. The mean value would be $\mathrm{E}(\Sigma_{zz}) = V\sigma_{zz}$, the variance would be $\mathrm{Var}(\Sigma_{zz}) = (V\sigma_{zz})^2/N$ and the coefficient of variation (defined as the ratio of the standard deviation to the mean) would be $CV = 1/\sqrt{N}$. This coefficient measures how far the set of random values of Σ_{zz} could spread out from their average value. For large values of N the CV tends to 0 and Σ_{zz} equals $V\sigma_{zz}$ in every realization of the experiment.

6 CONCLUSION

In discrete media, smooth stress fields are replaced by stochastic systems of force-chains. If these are transformed into equivalent stress fields, their values can be anticipated with stochastic models.

In particular, in Boussinesq-like problems the statistical distribution of the vertical extensive stress follows an exponential distribution whose parameter can be obtained from continuum based solutions.

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