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Recent advances in the application of discontinuity layout optimization to geotechnical analysis and design problems

Avancées récentes dans l'application de l'optimisation de la disposition des discontinuités aux problèmes d'analyse et de conception géotechniques

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ABSTRACT: Discontinuity layout optimization (DLO) is a powerful computational limit analysis procedure which provides a valuable and high accuracy alternative to traditional analysis techniques. For example, compared with finite element analysis methods, a key benefit of DLO is its inherent ability to straightforwardly model singularities and discrete slip-lines in the solution and present the results in a conventional slip-line format. In this paper recent advances are described, and results for a range of benchmark problems are presented. Advances described include: (i) modelling rotational failure mechanisms in frictional soils; (ii) geometrical optimization of solutions; (iii) design optimization of geotechnical structures.

RÉSUMÉ: L'optimisation du schéma de discontinuité (DLO) est une procédure d'analyse des limites de calcul puissante qui offre une alternative précieuse et de haute précision aux techniques d'analyse traditionnelles. Par exemple, par rapport aux méthodes d'analyse par éléments finis, l'un des principaux avantages de DLO est sa capacité inhérente à modéliser directement les singularités et les lignes de glissement discrètes dans la solution et à présenter les résultats dans un format de ligne de glissement classique. Dans cet article, nous décrivons les avancées récentes et présentons les résultats pour une série de problèmes de référence. Les progrès décrits incluent: (i) la modélisation des mécanismes de défaillance en rotation dans les sols soumis au frottement; (ii) optimisation géométrique des solutions; (iii) l'optimisation de la conception des structures géotechniques.

Keywords: Limit analyis; Discontinuity Layout Optimization; Geometry Optimization; Design optimization.

1 INTRODUCTION

Limit analysis provides a powerful means of directly modelling the collapse state of a geotechnical construction, allowing a margin of safety to be quickly established. While there exists a broad range of analytical limit analysis solutions for standard geotechnical problems, computational limit analysis methods have recently become available that can model a wide range of problem geometries. These methods include Finite Element Limit Analysis, FELA, (e.g. Lysmer, 1970; Sloan, 1988) and Discontinuity Layout Optimziation, DLO (Smith and Gilbert 2007).

In contrast to element based methods, DLO is formulated entirely in terms of discontinuities (e.g. slip-lines) and can directly identify upper bound mechanisms in the form of layouts of these discrete slip-line discontinuities, without the need for operator input. An example solution of a eccentrically loaded footing on clay is given in Figure 1. Such a solution illustrates the ability of DLO to directly model isolated sliplines which would otherwise require specialist meshes or mesh refinement in an FE approach.

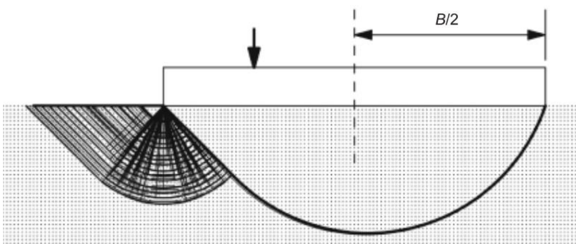


Figure 1. DLO analysis of an eccentrically loaded anchor on a cohesive medium (after Smith and Gilbert, 2013).

Since the original development of DLO in 2007, significant work has been undertaken to more fully realize its potential; for example, contributions by Clarke et al. (2010) addressed the modelling of soil reinforcement, Hawksbee et al. (2013) examined solution of 3D problems, Smith and Gilbert (2013, 2015) investigated modelling of rotational / translational mechanisms.

Smith et al. (2014) presented an overview of some of these advances in 2014. This paper reviews further recent advances, introduces some solutions to new example problems, and discusses potential future developments.

A brief description of the DLO procedure follows in Section 2, after which the recent advances are outlined.

2 GENERIC DLO PROBLEM FORMULATION

Stages in the DLO procedure are outlined schematically in Figure 2. This is shown for the plane strain translational case for simplicity, but can be extended to 3D and rotational cases. In the procedure, the plastic limit analysis problem is couched in terms of the potential discontinuities which inter-link nodes used to discretize the region under consideration. In the kinematic formulation compatibility at nodes is explicitly enforced. The critical layout of discontinuities is identified using optimization to minimise the energy dissipated in the mechanism (hence the term ‘discontinuity layout optimization’).

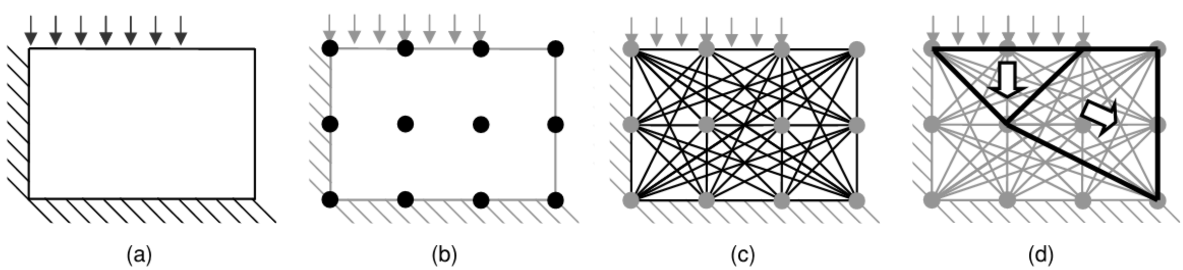


Figure 2. Stages in DLO procedure (after Gilbert et al. (2010)): (a) starting problem (surcharge applied to block of soil close to a vertical cut); (b) discretization of soil using nodes; (c) interconnection of nodes with potential discontinuities; (d) identification of critical subset of potential discontinuities using optimization (giving the layout of slip-lines in the critical failure mechanism).

The general kinematic formulation for 2D problems can be stated as follows (Smith and Gilbert 2007, 2013):

$$\min \lambda \mathbf{f}_L^T \mathbf{d} = -\mathbf{f}_\mathrm{D}^T \mathbf{d} + \mathbf{g}^T \mathbf{p} \quad (1)$$

subject to:

$$\mathbf{Bd} = \mathbf{0} \quad (2)$$

$$\mathbf{Np} - \mathbf{d} = 0 \quad (3)$$

$$\mathbf{f}_\mathrm{L}^T \mathbf{d} = 1 \quad (4)$$

$$\mathbf{p} \geq 0 \quad (5)$$

where λ is the unknown load factor at collapse, $\mathbf{f}_D$ and $\mathbf{f}_L$ are vectors containing respectively specified dead and live loads at discontinuities, **d** contains displacements along the discontinuities, **p** is a vector of plastic multipliers and **g** contains the corresponding dissipation coefficients. **B** is a suitable compatibility matrix containing direction cosines and **N** is a suitable flow matrix.

In qualitative terms, the method seeks a mechanism of collapse that is kinematically compatible (equation 2), obeys an associative flow rule (equation 3), while minimising external work and internal energy dissipation (rhs of equation 1). Equation 5 ensures that all internal work must be positive and equation 4 provides a unit live load work constraint such that the load factor λ is returned correctly.

3 ROTATIONAL ANALYSIS IN FRICTIONAL SOILS

3.1 Theory

The original paper on DLO (Smith & Gilbert 2007) considered only translational failure mechanisms. This was later extended to consider rotational/translational mechanisms in a cohesive (Tresca) media by Smith & Gilbert (2013, 15) and Smith et al. (2014) presented further solution examples.

The extension to rotational mechanisms requires that curved discontinuities inter-linking nodes be considered, rather than just straight-line discontinuities and that the displacements vector includes rotation (ω) as well as shear (*s*) and normal (*n*) displacments. For a cohesive material the curved discontinuities will be arcs of circles, which can be characterised by a subtending angle ψ.

Inclusion of rotations requires modification to both the compatiblity equations and the flow rule. Following the notation of Smith & Gilbert (2015), for a typical weightless cohesive plane-strain plastic analysis problem: $\mathbf{d}^T = \{\mathbf{d}^T_1, \mathbf{d}^T_2 \ldots \mathbf{d}^T_m\} = \{s_1, n_1, \omega_1, s_2, n_2, \omega_2 \ldots \omega_m\}$, where s_i, n_i and ω_i are the relative shear, normal, and rotational displacements at discontinuity *i*; $\mathbf{g}^T = \{c_1 l_1, c_l l_1, c_2 l_2, \ldots c_m l_m\}$, where c_i is the cohesive shear strength of discontinuity *i* and l_i is the discontinuity secant length.

Compatibility is enforced at the nodes. Thus the contribution of a given discontinuity *i* to the global compatibility constraint equation (2) can be written as:

$$\mathbf{B}_i \mathbf{d}_i = \begin{bmatrix} \alpha_i & -\beta_i & 0.5 l_i \beta_i \\ \beta_i & \alpha_i & -0.5 l_i \alpha_i \\ 0 & 0 & 1 \\ -\alpha_i & \beta_i & 0.5 l_i \beta_i \\ -\beta_i & -\alpha_i & -0.5 l_i \alpha_i \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} s_i \\ n_i \\ \omega_i \end{bmatrix} \quad (6)$$

where α_i and β_i are respectively *x*-axis and *y*-axis direction cosines for discontinuity *i*.

For a general cohesive-frictional soil, the local plastic flow constraint for discontinuity *i* (equation 3) can be written as:

$$\mathbf{N}_i \mathbf{p}_i - \mathbf{d}_i = \begin{bmatrix} A_i \tan\phi_i & -A_i \tan\phi_i \\ \tan\phi_i & \tan\phi_i \\ \mathrm{B}_i \tan\phi_i & -B_i \tan\phi_i \end{bmatrix} \begin{bmatrix} p_i^1 \\ p_i^2 \end{bmatrix} - \begin{bmatrix} s_i \\ n_i \\ \omega_i \end{bmatrix} = 0 \quad (7)$$

where for a cohesive soil: $\phi_i = 0$, $A_i = \frac{\sin\psi_i}{\psi_i}$ and $B_i = \frac{4\sin^2\frac{\psi_i}{2}}{\psi_i l_i}$. For a frictional soil, the curved discontinuities become log-spirals, and A and B become functions of ϕ and ψ. There is insufficient space to detail the full formulation in this paper which will be presented in a forthcoming journal paper.

3.2 Anchor analysis

Initial work using the full frictional-rotation formulation has examined the classic anchor/trapdoor uplift problem. Figure 3 shows the half-space analysis of a anchor for ϕ = 15°. The solution can be seen to be closely similar to the stress characteristics solution derived in the lower bound solution by Smith (1998).

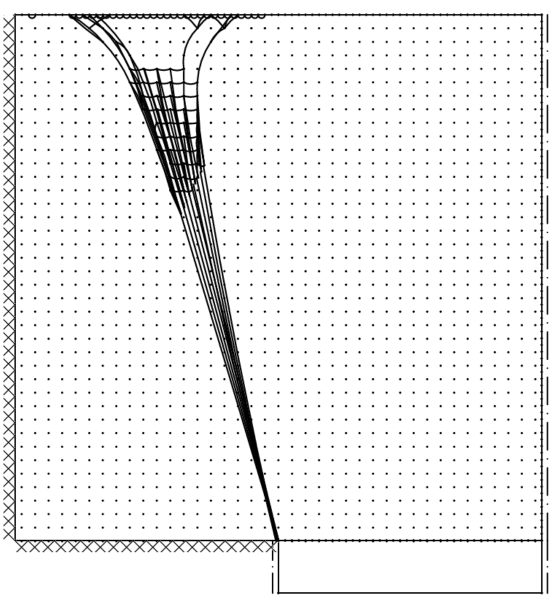


Figure 3. DLO analysis of an anchor uplift problem for a c=0, ϕ = 15° soil possessing self weight.

4 GEOMETRY OPTIMIZATION

The conventional DLO procedure is based on a fixed initial distribution of nodes as shown in Figure 2. In general, the accurancy of the solution increases as the nodal density increases since this provides a richer set of potential sliplines from which to construct a solution. A robust algorithm to systematically and efficiently add or move nodes to areas that would increase accuracy is an ongoing research goal. Largely heuristic based approaches have been suggested by e.g. Crumpton et al. (2014) with varying degrees of success.

Geometry optimization, GO (He et al., 2015) is a generally robust post-processing approach which takes a conventional DLO solution as its starting point and optimizes this through sequential movements of the nodes.

In GO, in addition to the original DLO variables (the displacements *s*, *n*, ω in **d** and plastic multipliers in **p**), nodal positions at the ends of each discontinuity (x_A, y_A), (x_B, y_B) are also considered as optimization variables.

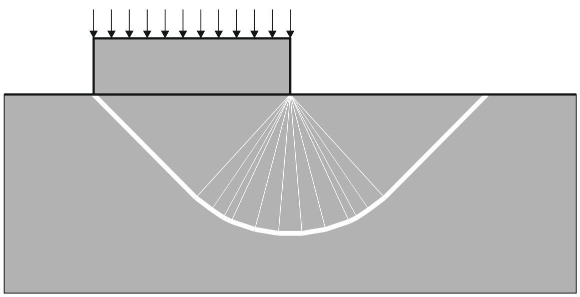
In addition, with respect to the original optimization formulation, the objective function (equation 1), nodal compatibility constraint (equation 2), and unit displacement constraint (equation 4) now become non-linear, thus leading to a non-linear programming (NLP) problem. To solve this problem efficiently, the first and second derivatives of the objective function and constraints can be derived analytically, and efficient non-linear optimization packages can be utilized.

Due to the non-linearity and non-convexity of the problem, it is necessary to impose move limits on the nodes and conduct a sequential solution procedure of a series of small nodal movements. If nodes migrate towards each other then a node merging procedure is adopted. Full details of the general approach can be found in He and Gilbert (2015, 2016).

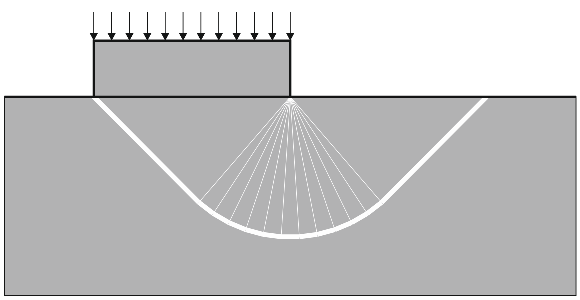
Figure 4(a) shows the solution of the classic Prandtl punch mechanism by DLO. Figure 4(b) gives the same solution following the GO process. The solution has improved in accuracy by 0.06%, and the mechanism has become significantly more uniform.

Figures 5(a) and (b) present a similar comparison for the problem of the squeezing of a layer of clay with an improvement in accuracy of 0.07%.

While this small gain in accuracy may not be significant in practical engineering terms, the same improvement using conventional DLO would require a significantly increased nodal density and a significantly longer solve time compared to the combined DLO-GO process. In addition, for more complex cases, the gain in accuracy can be more significant.

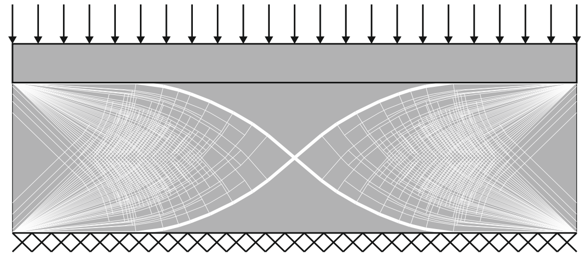


(a)

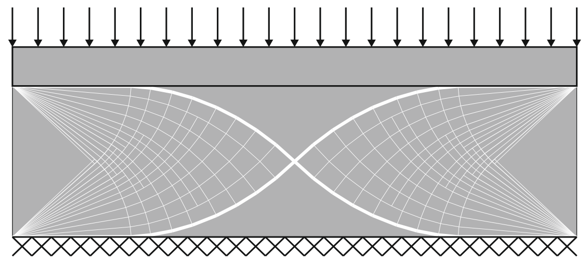


(b)

*Figure 4. (a) DLO analysis of the Prandtl punch 101*51 nodes (solution:5.1493, error: 0.15%) (b) analysis following an additional GO stage (solution: 5.1462, error 0.09%). True solution: 5.1416.*



(a)



(b)

*Figure 5. (a) symmetric half space DLO analysis of the squeezing of a confined layer of clay, thickness L and width 4L, 151 * 41 nodes (solution: 3.5173, error 0.14%) (b) analysis following an additional GO stage (solution: 3.5148, error 0.07%). True solution: 3.5124 (Graczykowski and Lewinski, 2010).*

The final mechanism is also much 'cleaner' and easier to interpret, being much closer to the style of solution given by the classic method of characteristics (Sokolovskii, 1965), familiar to many engineers. This is illustrated particularly well in Figure 5(b).

Further development of the technique aims to generate solutions that are essentially equivalent to a method of characteristics solution and to then exploit this as the basis of a lower bound solution, thus fully bracketing the actual solution.

This would then overcome a key challenge for the method of characteristics in that for complex problems it is difficult to determine boundary conditions and to delineate non-yielding areas.

5 DESIGN OPTIMIZATION

The original DLO formulation utilises optimization to determine the critical collaspe mechanism. Recent work by González-Castejón and Smith (2018) has demonstrated how the optimzation equations can be reformulated to move from an *analysis* optimization procedure to a *design* optimization procedure.

In the example presented in that paper, the process is set up to determine the optimal strength of reinforcement for a slope stabilisation problem, a process termed Reinforcement Strength Optimization (RSO).

RSO adopts a revised formulation based on the previously described DLO equations, extended to model reinforcement as described by Clarke et al. (2013) or González-Castejón and Smith (2018), and able to determine the maximum global reduction factor on the tensile strength that can be applied to a pre-determined extent of reinforcement and that still maintains stability. This is formulated as follows:

$$\min \lambda \mathbf{g}_r^T \mathbf{p}_\mathrm{r} = -\mathbf{f}_\mathrm{D}^T \mathbf{d} + \mathbf{g}_s^T \mathbf{p}_\mathrm{s} \quad (6)$$

subject to:

$$\mathbf{Bd} = \mathbf{0} \quad (7)$$

$$\mathbf{Np} - \mathbf{d} = 0 \quad (8)$$

$$\mathbf{g}_\mathbf{r}^T \mathbf{p}_\mathrm{r} = 1 \quad (9)$$

$$\mathbf{p}_\mathrm{s}, \mathbf{p}_\mathrm{r} \geq 0 \quad (10)$$

Where the subscripts *s* and *r* represents the soil and the reinforcement respectively, $\mathbf{p} = \{\mathbf{p}_s, \mathbf{p}_r\}^T$ and $\mathbf{f}$ now represents all loading in the system (there is no longer a distinction between live and dead loads). g_s is now equal to *cl* for a discontinuity in the soil and αcl for a discontinuity running along the edge of the reinforcement where α is the soil/reinforcement interface factor, and g_r denotes the tensile strength of the reinforcement.

In this case the linear programming solution determines a uniform factor applied to the reinforcement tensile capacity needed in order to carry a given external load. Figure 6 presents a solution for a reinforcement configuration designed using the UK Highways Agency manual Design Methods for the Reinforcement of Highway Slopes by Reinforced Soil and Soil Nailing Techniques (HA 68/94 1994) and predicts an overdesign of ~ 40% in terms of reinforcement strength.

Work is ongoing to extend this process to determine optimal layout and strength for reinforcement. The full formulation will be presented in a forthcoming journal paper and examples presented at ECSMGE2019.

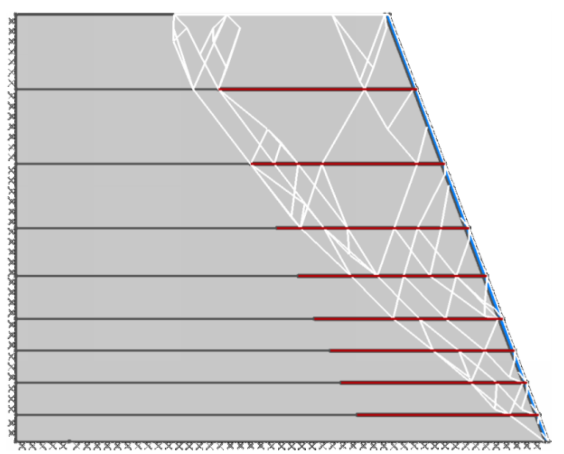


Figure 6. Reinforcement layout for a 70° slope, with reinforcement strength 14.4 kN/m and a variable spacing using the HA 68/94 guidelines. Mechanism shown is from an RSO solution which predicts a required reinforcement strength of 10.2 kN/m. (following González-Castejón and Smith, 2018).

6 CONCLUSIONS

Discontinuity layout optimization (DLO) provides a highly efficient tool for geotechnical engineers, effectively automating the traditional 'upper bound' hand analysis method. The method continues to be developed in a variety of directions.

Three key current extensions have been reported: rotational analysis of cohesive / frictional media, geometry optimization which is able to generate failure mechanisms of signifcant clarity and precision, and design optimization illustrated by the optimization of the reinforcement within a reinforced slope.

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